

EMBEDDING PROBLEMS FOR OPEN SUBGROUPS OF THE FUNDAMENTAL GROUP

MANISH KUMAR

ABSTRACT. Let C be a smooth irreducible affine curve over an algebraically closed field of positive characteristic and let $\pi_1(C)$ be its fundamental group. We study various embedding problems for $\pi_1(C)$ and its subgroups.

1. INTRODUCTION

Let C be a smooth irreducible affine curve over an algebraically closed field k of characteristic $p > 0$. The existence of wild ramification causes the structure of the étale fundamental group $\pi_1(C)$ to be complicated. There has been some attempts to understand this group. One step towards this goal was Raynaud's [Ray] and Harbater's [Ha94] proof of the Abhyankar's conjecture which states that a finite group G is a quotient of $\pi_1(C)$ if and only if the maximal prime-to- p quotient $G/p(G)$ of G is generated by $2g + r - 1$ elements where g is the genus of the smooth completion of C and r is the number of points in the boundary.

Though this gives a complete description of the finite quotients of $\pi_1(C)$, it does not say how these groups fit together in the inverse system for $\pi_1(C)$. A possible way to understand this is by analyzing which finite embedding problems for $\pi_1(C)$ have a solution (see the "Notation" subsection at the end of this introduction for definition). One crucial result in this direction is by Pop [Pop] (and independently proved by Harbater as well [Ha03, Theorem 5.3.4]) which says that given an embedding problem $\mathcal{E} = (\alpha : \Pi \rightarrow G, \phi : \Gamma \rightarrow G)$ where $H = \ker(\phi)$ is a quasi- p group, there is a proper solution $\lambda : \Pi \twoheadrightarrow \Gamma$ to \mathcal{E} . This clearly strengthens Raynaud's and Harbater's result. When H is not a quasi- p group then it is clear from the Abhyankar's conjecture that the above embedding problem may not have a solution. But there are finite index open subgroups of $\pi_1(C)$ for which these embedding problems have a solution.

Definition 1. Given an embedding problem $\mathcal{E} = (\alpha : \pi_1(C) \rightarrow G, \phi : \Gamma \rightarrow G)$ and a finite index subgroup Π of $\pi_1(C)$, we say the embedding problem *restricts* to Π if $\alpha(\Pi) = G$. Moreover if the restricted embedding problem has a solution then we say Π is *effective* for the embedding problem \mathcal{E} .

In [HS09], it was shown that given any finite embedding problem for $\pi_1(C)$ there exist a finite index effective subgroup for the embedding problem. In [BK11, Theorem 1.3] it was shown that it is even possible to find an index p subgroup of $\pi_1(C)$ which is effective for the embedding problem. Our objective is to find some necessary and some sufficient conditions for a subgroup of $\pi_1(C)$ to be an effective subgroup for a given embedding problem.

Suppose Π is a finite index subgroup of $\pi_1(C)$. This corresponds to a cover $D \rightarrow C$. Let $Z \rightarrow X$ be the corresponding morphism between their smooth completions

and let n_D denote $2g_Z + r_D - 1$. A necessary condition for Π to be effective for the embedding problem $\mathcal{E} = (\alpha : \pi_1(C) \rightarrow G, \phi : \Gamma \rightarrow G)$ is that \mathcal{E} restricts to Π and the rank of $\Gamma/p(\Gamma)$ is at most n_D . Indeed Π is effective implies there is a surjection from $\pi_1(D) = \Pi \twoheadrightarrow \Gamma$. Hence there is a surjection from prime-to- p part of $\pi_1(D)$ to the maximal prime-to- p quotient of Γ , denoted $\Gamma/p(\Gamma)$. So $\Gamma/p(\Gamma)$ is generated by n_D elements. Also, [HS09, Theorem 5] can be rephrased in terms of sufficient conditions for $\Pi = \pi_1(D)$ to be an effective subgroup of $\pi_1(C)$.

Proposition 2. *Let $C \subset X$, Π , \mathcal{E} , and $D \subset Z$ be as above. Let $\Psi_X : V_X \rightarrow X$ be the G -cover corresponding to the embedding problem \mathcal{E} and assume \mathcal{E} restricts to Π . Let $\tilde{\Psi} : V_X \times_X Z \rightarrow Z$ be the pull-back of Ψ_X . Then Π is effective if the number of points in $Z \setminus D$ where the morphism $\tilde{\Psi}$ is not branched, is at least the relative rank of $\ker(\phi : \Gamma \rightarrow G)$ in Γ .*

The subset of $Z \setminus D$ for which the morphism Ψ is unramified can be thought of as points available for branching in any cover dominating Ψ . In the theorem above, having sufficiently many available branch points allows the embedding problem to be solved. A natural question is can the condition on the number of such potential branch points in $Z \setminus D$ be relaxed? For example, could it be replaced by a condition on the genus of Z or n_D ? It is also worth noting that in the above proposition the degree of the cover $Z \rightarrow X$ must be large to ensure that the number of points in $Z \setminus D$ where the morphism $\tilde{\Psi}$ is not branched is sufficiently large. Hence the index of Π in $\pi_1(C)$ is also large.

The two main results of this paper, Theorem 17 and Proposition 21, use genus and branch points to obtain effective subgroups. Let H be the kernel of the homomorphism $\phi : \Gamma \rightarrow G$ from the embedding problem $\mathcal{E} = (\alpha : \pi_1(C) \rightarrow G, \phi : \Gamma \rightarrow G)$, and let $H/p(H)$ be the maximal prime to p quotient of H . The main tool in this paper is Theorem 14 which investigates the relationship between solving embedding problems for $\pi_1(C)$ and the relative rank of $H/p(H)$ in $\Gamma/p(H)$. Roughly speaking, it shows that if the G -Galois cover $\Psi_X : V_X \rightarrow X$ corresponding to α has a deformation which is sufficiently degenerate (in terms of having many components that are trivial G covers), then there exists a proper solution to \mathcal{E} . Theorem 17 restricts to the case that C is the affine line and Π has index p in $\pi_1(\mathbb{A}_k^1)$. It uses degenerations and Theorem 14 to show that if the curve Z in the $\mathbb{Z}/p\mathbb{Z}$ -cover $Z \rightarrow \mathbb{P}_k^1$ corresponding to $\Pi \subset \pi_1(\mathbb{A}_k^1)$ has genus g_Z greater than the relative rank of (H/pH) in $\Gamma/p(H)$ and a technical condition which holds for most values of g_Z (see Corollary 18, Corollary 19 and Remark 20) then Π is effective for \mathcal{E} . In particular these results provide sufficient conditions for subgroups of $\pi_1(C)$ to be effective. In Section 7 some of the results proved for affine line case are generalized to general curves though the conclusions obtained are slightly weaker (Proposition 21 and Corollary 23).

The techniques involved in proving these results include formal patching and deformations within families of covers. Each result requires the construction of a Galois cover of a degenerate curve. Then a formal patching argument is used to obtain a cover of smooth curves over a complete local ring. Finally, a deformation argument similar to [HS09, Proposition 4] is used to ensure that the original embedding problem \mathcal{E} restricts the subgroup Π of $\pi_1(C)$ that is obtained in the construction. The paper is organized as follows: At the end of the Introduction there is a list of notation. Section 2 looks at some of the group theoretic properties and examples of effective groups. Section 3 defines deformations and degenerations

of covers and uses a formal patching result of Harbater to solve the embedding problems in the presence of a sufficiently degenerate deformation of the original cover. Section 4 proves a globalization and specialization result using the Lefschetz type principle and Abhyankar's lemma. Section 5 applies the formal patching and Lefschetz-Abhyankar result to obtain Theorem 14. Finally in Section 6, Theorem 17 is proved, and in Section 7, Proposition 21 and Corollary 23 is proved.

Notation. If G is a finite group and p is a prime number, let $p(G)$ denote the subgroup of G generated by its p -subgroups. This is a characteristic subgroup of G , and $G/p(G)$ is the maximal prime-to- p quotient of G . A finite group G is called *quasi- p* if $G = p(G)$.

Let Γ be any finite group and let H be a subgroup of Γ . A subset $S \subset H$ will be called a *relative generating set* for H in Γ if for every subset $T \subset \Gamma$ such that $H \cup T$ generates Γ , the subset $S \cup T$ also generates Γ . We define *the relative rank of H in Γ* to be the smallest non-negative integer $\mu := \text{rank}_\Gamma(H)$ such that there is a relative generating set for H in Γ consisting of μ elements. Every generating set for H is a relative generating set, so $0 \leq \text{rank}_\Gamma(H) \leq \text{rank}(H)$. Also, $\text{rank}_\Gamma(H) = \text{rank}(H)$ if H is trivial or $H = \Gamma$, while $\text{rank}_\Gamma(H) = 0$ if and only if H is contained in the Frattini subgroup $\Phi(\Gamma)$ of Γ [Ha99, p. 122].

A *finite embedding problem* \mathcal{E} for a group Π is a pair of surjections $(\alpha : \Pi \rightarrow G, \phi : \Gamma \rightarrow G)$, where Γ and G are finite groups. If $H = \ker(\phi)$, the embedding problem \mathcal{E} can be summarized by

$$\begin{array}{ccccccc} & & & & \Pi & & \\ & & & & \downarrow \alpha & & \\ 1 & \longrightarrow & H & \longrightarrow & \Gamma & \xrightarrow{\phi} & G \longrightarrow 1 \\ & & & & & & \downarrow \\ & & & & & & 1 \end{array}$$

A *weak solution* to \mathcal{E} is a homomorphism $\gamma : \Pi \rightarrow \Gamma$ such that $\phi \circ \gamma = \alpha$. We call γ a *proper solution* to \mathcal{E} if in addition it is surjective.

Remark 3. Let Π be a profinite group and let $\mathcal{E} = (\alpha : \Pi \rightarrow G, \phi : \Gamma \rightarrow G)$ be a finite embedding problem for Π . Suppose that the epimorphism $\alpha : \Pi \rightarrow G$ factors as $r\alpha'$, where $\alpha' : \Pi \rightarrow G'$ and $r : G' \rightarrow G$ are epimorphisms, for some finite group G' . We consider the *induced embedding problem* $\mathcal{E}_{\alpha'} = (\alpha' : \Pi \rightarrow G', \phi' : \Gamma' \rightarrow G')$ by taking $\Gamma' = \Gamma \times_G G'$ and letting $\phi' : \Gamma' \rightarrow G'$ be the second projection map. Here ϕ' is surjective because ϕ is; and so $\mathcal{E}_{\alpha'}$ is a finite embedding problem. Here \mathcal{E} and $\mathcal{E}_{\alpha'}$ have isomorphic kernels; indeed $\ker(\phi') = \ker(\phi) \times 1 \subset \Gamma \times_G G' = \Gamma'$. Note that the first projection map $q : \Gamma' = \Gamma \times_G G' \rightarrow \Gamma$ is surjective since $r : G' \rightarrow G$ is surjective; and $\phi q = r\phi'$.

In this situation, every proper solution $\lambda' : \Pi \rightarrow \Gamma'$ of $\mathcal{E}_{\alpha'}$ induces a proper solution $\lambda := q\lambda' : \Pi \rightarrow \Gamma$ of \mathcal{E} ; viz. $\phi\lambda = \phi q\lambda' = r\phi'\lambda' = r\alpha' = \alpha$, and λ is surjective because q and λ' are. So we obtain a map $\text{PS}(\mathcal{E}_{\alpha'}) \rightarrow \text{PS}(\mathcal{E})$, where PS denotes the set of proper solutions to the embedding problem.

In this paper we consider curves over an algebraically closed field k of characteristic $p > 0$. A *cover* of k -curves is a morphism $\Phi : D \rightarrow C$ of smooth connected k -curves that is finite and generically separable. If $\Phi : D \rightarrow C$ is a cover, its *Galois*

group $\text{Gal}(D/C)$ is the group of k -automorphisms σ of D satisfying $\Phi \circ \sigma = \Phi$. If G is a finite group, then a G -Galois cover is a cover $\Phi : D \rightarrow C$ together with an inclusion $\rho : G \hookrightarrow \text{Gal}(D/C)$ such that G acts simply transitively on a generic geometric fiber of $\Phi : D \rightarrow C$. If we fix a geometric point of C to be a base point, then the pointed G -Galois étale covers of C correspond bijectively to the surjections $\alpha : \pi_1(C) \rightarrow G$, where $\pi_1(C)$ is the algebraic fundamental group of C with the chosen geometric point as the base point. The proper solutions to an embedding problem $\mathcal{E} = (\alpha : \pi_1(C) \rightarrow G, \phi : \Gamma \rightarrow G)$ for $\pi_1(C)$ then are in bijection to the pointed Γ -Galois covers $E \rightarrow C$ that dominate the pointed G -Galois cover $\Phi : D \rightarrow C$ corresponding to α . In the case that X is the smooth completion of the affine k -curve C , denote by g_X the genus of X , and define $r_C = \#(X - C)$ and $n_C = 2g_X + r_C - 1$.

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2. GROUP THEORY RESULTS AND EXAMPLES OF EFFECTIVE SUBGROUPS

In this section we analyze when a subgroup of an effective subgroup for an embedding problem is effective using some group theory and Galois theory.

We start with an embedding problem $\mathcal{E}(1)$ for Π .

$$(1) \quad \begin{array}{ccccccc} & & & & \Pi & & \\ & & & & \downarrow \alpha & & \\ & & & \swarrow \phi & & & \\ 1 & \longrightarrow & H & \longrightarrow & \Gamma & \longrightarrow & G \longrightarrow 1 \\ & & & & & & \downarrow \\ & & & & & & 1 \end{array}$$

The following remark is easy to see.

Remark 4. Suppose Π_1 is an effective subgroup for the embedding problem $\mathcal{E}(1)$ and Π_2 is a finite index subgroup of Π_1 . Suppose for some solution ψ of the embedding problem $\mathcal{E}(1)$ restricted to Π_1 , the index of $\ker(\psi) \cap \Pi_2$ in Π_2 is $|\Gamma|$ then Π_2 is also an effective subgroup for $\mathcal{E}(1)$. Note that $\ker(\psi|_{\Pi_2}) = \ker(\psi) \cap \Pi_2$ has index $|\Gamma|$ in Π_2 . Hence $\psi|_{\Pi_2}$ indeed surjects onto Γ .

Corollary 5. Suppose Π_1 is an effective subgroup for the embedding problem $\mathcal{E}(1)$ and Π_2 is a finite index subgroup of Π_1 such that the embedding problem $\mathcal{E}(1)$ restricts to Π_2 and $[\Pi_1 : \Pi_2]$ is coprime to $|H|$ then Π_2 is also effective for $\mathcal{E}(1)$.

Proof. Let ψ be a solution to the embedding problem $\mathcal{E}(1)$ restricted to Π_1 . Note that $\ker(\psi) \subset \ker(\alpha|_{\Pi_1})$ and $[\ker(\alpha|_{\Pi_1}) : \ker(\psi)] = |H|$. Since $[\Pi_1 : \Pi_2]$ is coprime to $|H|$, $[\ker(\alpha|_{\Pi_1}) : \ker(\alpha|_{\Pi_2})]$ is also coprime to $|H|$. So the index $[\ker(\alpha|_{\Pi_2}) : \ker(\psi) \cap \ker(\alpha|_{\Pi_2})] = |H|$. The embedding problem restricts to Π_2 means that $[\Pi_2 : \ker(\alpha|_{\Pi_2})] = |G|$. So we obtain that $[\Pi_2 : \ker(\psi) \cap \ker(\alpha|_{\Pi_2})] = |G||H| = |\Gamma|$. Also note that $\ker(\psi) \cap \ker(\alpha|_{\Pi_2}) = \ker(\psi) \cap \ker(\alpha|_{\Pi_1}) \cap \Pi_2 = \ker(\psi) \cap \Pi_2$. Hence the result is obtained from the previous remark. \square

In the above corollary the hypothesis guaranteed that if Π_1 is an effective subgroup for the given embedding problem and ψ is a solution of the embedding problem restricted to Π_1 then $\psi|_{\Pi_2}$ is a solution to the embedding problem restricted to Π_2 . Hence Π_2 is also an effective subgroup. But this does not hold unconditionally as the following examples show.

Example. Let Π be the absolute Galois group of the reals, let $\Gamma = H = \mathbb{Z}/2\mathbb{Z}$, and let $G = 1$. Then the given embedding problem has a proper solution (the complex numbers). So $\Pi_1 = \Pi$ itself is effective. But if we pull back from \mathbb{R} to \mathbb{C} (corresponding to taking the trivial subgroup Π_2 of Π) then the embedding problem (which has trivial cokernel) restricts to Π_2 . But it no longer has a proper solution. Note here $|H| = [\Pi : \Pi_2] = 2$.

On the contrary, in geometric setting, even if the given solution does not pull back to a proper solution, there might be some other proper solution over the pullback.

Example. Let C be the affine x -line minus 0 in characteristic 0, let $\Pi = \pi_1(C)$, $\Gamma = \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/3\mathbb{Z}$, $H = \mathbb{Z}/3\mathbb{Z}$, and $G = \mathbb{Z}/2\mathbb{Z}$. Then there is a G -cover C_1 of C given by $y^2 = x$. Over that there's a proper solution D to the $\mathcal{E}(1)$, given by $z^3 = y$. So again $\Pi_1 = \Pi$ is an effective subgroup of Π for $\mathcal{E}(1)$. Here D is the fiber product of C_1 with the curve C_2 given by $w^3 = x$ (where $w = z^2$, and $z = y/w$). Now pull back everything by the degree 3 cover C_2 of C . This is linearly disjoint from C_1 , so $\mathcal{E}(1)$ restricts to the subgroup $\pi_1(C_2)$ of Π . But the degree is not relatively prime to $|H|$, the cover C_2 is not linearly disjoint from D , and the solution to the given $\mathcal{E}(1)$ does not restrict to a proper solution to $\mathcal{E}(1)$ restricted to $\pi_1(C_2)$. But $\mathcal{E}(1)$ restricted to $\pi_1(C_2)$ does have a proper solution, given by $v^3 = z$.

3. FORMAL PATCHING RESULTS

In this section we develop some formal patching results which are used in later sections to find solutions to various embedding problems. Proposition 12 is the main result of this section and one of the main technical results of this paper.

Notation. Given a scheme X , denote by $\mathcal{M}(X)$ the category of coherent sheaves of \mathcal{O}_X -modules, $\mathcal{AM}(X)$ the category of coherent sheaves of \mathcal{O}_X -algebras and $\mathcal{SM}(X)$ the subcategory of $\mathcal{M}(X)$ for which the sheaves of algebras are generically separable and locally free. Given a finite group G denote by $GM(X)$ the category of generically separable coherent locally free sheaves of \mathcal{O}_X -algebras S together with a G -action which is transitive on the geometric generic fibers of $\mathrm{Spec}_{\mathcal{O}_X}(S) \rightarrow X$. Given categories $\mathcal{A}, \mathcal{B}, \mathcal{C}$ and functors $\mathfrak{F} : \mathcal{A} \rightarrow \mathcal{C}$ and $\mathfrak{G} : \mathcal{B} \rightarrow \mathcal{C}$, denote by $\mathcal{A} \times_{\mathcal{C}} \mathcal{B}$ the associated fiber category.

The following result is due to Harbater [Ha03, Theorem 3.2.12].

Theorem 6. *Let (A, \mathfrak{p}) be a complete local ring and T a proper A -scheme. Let $\{\tau_1, \dots, \tau_N\}$ be a finite set of closed points of T and $U = T \setminus \{\tau_1, \dots, \tau_N\}$. Denote by $\hat{\mathcal{O}}_{T, \tau_i}$ the completion of the local ring \mathcal{O}_{T, τ_i} and let $T_i = \mathrm{Spec}(\hat{\mathcal{O}}_{T, \tau_i})$. Let U^* be the \mathfrak{p} -adic completion of U and \mathcal{K}_i the \mathfrak{p} -adic completion of $T_i \setminus \{\tau_i\}$. Then the base change functor*

$$\mathcal{M}(T) \rightarrow \mathcal{M}(U^*) \times_{\mathcal{M}(\bigcup_{i=1}^N \mathcal{K}_i)} \mathcal{M}\left(\bigcup_{i=1}^N T_i\right)$$

is an equivalence of categories. The same remains true if we replace \mathcal{M} by \mathcal{AM} , \mathcal{SM} or \mathcal{GM} for a fixed finite group G .

Next we state a lemma useful for putting covers into a situation where the theorem above holds. Its proof is just an application of a generalization of the Noether Normalization Lemma.

Lemma 7. *Let C be an affine k -curve and X be its smooth projective completion. Then there exists a finite morphism $\Theta_X : X \rightarrow \mathbb{P}_x^1$ such that Θ_X is étale at $\Theta_X^{-1}(\{x = 0\})$ and such that $\Theta_X^{-1}(\{x = \infty\}) = X \setminus C$.*

Proof. By a stronger version of Noether normalization (cf. [Eis, Corollary 16.18]), there exist finite proper generically separable k -morphisms from $C \rightarrow \mathbb{A}_x^1$ where \mathbb{A}_x^1 is the affine k -line with local coordinate x . The branch locus of this morphism is of codimension 1, and hence it is étale away from finitely many points. By translation we may assume that $x = 0$ is not a branch point of $C \rightarrow \mathbb{A}_x^1$. This morphism extends to a finite proper morphism $\Theta_X : X \rightarrow \mathbb{P}_x^1$. Note that Θ_X is étale at $\Theta_X^{-1}(\{x = 0\})$ and that $\Theta_X^{-1}(\{x = \infty\}) = X \setminus C$. \square

The patching results will be applied to covers of reducible curves where there are sufficiently many components of the base over which the cover is trivial. First, we need some terminology.

Definition 8. Let $\Phi : V \rightarrow X$ be a G -cover of smooth irreducible projective curves over k . Assume X has r marked points and Φ is étale away from these r points. We say that a G -cover of connected projective curves $\Phi' : V' \rightarrow X'$ with r marked points on X' is a *deformation* of $V \rightarrow X$ if there exist a smooth irreducible k -scheme S , a cover of S -curves $\Phi_S : V_S \rightarrow X_S$, r sections $p_1, \dots, p_r : S \rightarrow X_S$ and closed points s and s' in S such that the following three conditions hold.

- (1) Φ_S induces Φ and Φ' at s and s' respectively.
- (2) $\{p_1(s), \dots, p_r(s)\}$ and $\{p_1(s'), \dots, p_r(s')\}$ are the marked points of X and X' respectively.
- (3) Φ_S is a G -cover étale away from the sections p_1, \dots, p_r .

Notation. Given a $\Phi : V \rightarrow X$ and a deformation $\Phi' : V' \rightarrow X'$, we will call $(S; \Phi_S : V_S \rightarrow X_S; p_1, \dots, p_r; s; s')$ the associated data of the deformation.

Definition 9. The deformation $V' \rightarrow X'$ will be called a *SNC deformation* if all irreducible components of X' are smooth and they intersect transversely. Moreover, it will be called a *degeneration* if it is SNC and for some irreducible component X_1 of X' , the restriction of the G -cover to X_1 is induced from a trivial cover, i.e. $V' \times_{X'} X_1 \cong \text{Ind}_e^G X_1$ as G -covers of X_1 . In this situation X_1 will be called a trivial component of Φ' .

Remark 10. Let $\Phi : V_X \rightarrow X$ be a G -cover. Suppose Φ has a deformation $\Phi' : V_{X'} \rightarrow X'$ with associated data $(S; \Phi_S : V_S \rightarrow X_S; p_1, \dots, p_r; s; s')$. By taking a smooth irreducible k -curve in S passing through s and s' and pulling back $\Phi_S : V_S \rightarrow X_S$ to this curve, we may assume S is a smooth k -curve. In other words, if Φ' is a deformation of Φ then we may assume Φ can be deformed to Φ' along a smooth curve.

Example. Let $\Phi : V_X \rightarrow X$ be a G -cover of irreducible smooth projective curves. Let τ be a closed point in X . Let $S = \mathbb{A}_t^1$ and X_S be the blowup of $(\tau, t = 0)$

in $X \times S$ and X' be the total transform of the zero locus of $t = 0$ in $X \times S$. Note that X' has two irreducible components, a copy of X and the exceptional divisor isomorphic to \mathbb{P}^1 intersecting at τ . One can obtain a G -cover $\Phi_S : V_{X_S} \rightarrow X_S$ obtained by pullback along $X_S \rightarrow X \times S \rightarrow X$. The fiber over $t = 0$ induces a G -cover $\Phi' : V_{X'} \rightarrow X'$. The exceptional divisor \mathbb{P}^1 is the trivial component of Φ' .

Lemma 11. *Let $\Phi : V \rightarrow S$ be a flat family of reduced projective irreducible curves in which S is a smooth connected variety and V is a normal variety. Suppose for every point $s \in S$ the normalization of the fiber V_s has the same genus. Then for every closed point $s \in S$ the fiber V_s is smooth.*

Proof. Let η be the generic point of S . Note that V_η being a localization of V is normal. Hence there exist a nonempty open subset U of S such that for all closed point $s \in U$, V_s is a normal k -curve and hence smooth. Let g be the genus of such a curve. Suppose there exist $s' \in S$ a closed point such that $V_{s'}$ is singular. Then the arithmetic genus of $V_{s'}$ is g , since Φ is a flat family. But $V_{s'}$ is singular so the geometric genus $p_g(V_{s'}) < g$. But $p_g(V_{s'})$ is same as the genus of the normalization of $V_{s'}$. This contradicts the hypothesis that the normalization of every fiber has the same genus. \square

Proposition 12. *Let Γ be a finite group. Let G be a subgroup of Γ and let H_1, \dots, H_m be subgroups of Γ of order prime-to- p . Assume that G, H_1, \dots, H_m generate Γ . Let C be an affine k curve with smooth completion X . Let $\Phi : V_X \rightarrow X$ be a G -Galois cover of X étale over C . Let $B_X = X - C$, $r = \#(B_X)$, and consider the set B_X to be r marked points on X . Suppose Φ has a degeneration $\Phi' : V'_X \rightarrow X'$ with an associated data $(S; \Phi_S : V_S \rightarrow X_S; p_1, \dots, p_r; s; s')$ such that S is a smooth curve and X_1, \dots, X_m are the trivial components of X' . Let B be the union of the images of the sections $p_1, \dots, p_r : S \rightarrow X_S$ and let r_i be the number of smooth marked points of X' lying on X_i for $i = 1, \dots, m$. Further assume that for each $1 \leq i \leq m$, there exist an H_i -cover $\Phi_{X_i} : W_{X_i} \rightarrow X_i$ étale away from these r_i points. Let $T = X_S \times_S \hat{S}_{s'}$ where $\hat{S}_{s'}$ is the completion of S at s' and $\Phi_T : V_T \rightarrow T$ be the pullback of Φ_S to T . Then there exists a Γ -cover $\Psi : W \rightarrow T$ such that Ψ is étale away from $B \times_{X_S} T$. Moreover, if $\Gamma = H \rtimes G$ and $H_1, \dots, H_m \leq H$ then Ψ dominates the G -cover Φ_T .*

Proof. By Remark 10 we may assume that there exist an associated data $(S; \Phi_S : V_S \rightarrow X_S; p_1, \dots, p_r; s; s')$ for the degeneration of Φ to Φ' such that S is a smooth curve. Let t be the local coordinate of S at s' . Then $\hat{S}_{s'} = \text{Spec}(k[[t]])$. By construction, the closed fiber of T is X' . Let X_0 be the closure of $X' \setminus \bigcup_{i=1}^m X_i$ in X' . So X_0 is made up of the nontrivial components of Φ' . Since Φ' is a degeneration of Φ it is a SNC deformation, so all irreducible components of X' are smooth and they intersect transversely. Let τ_1, \dots, τ_N be the closed points of T where X_i and X_j intersect for some $0 \leq i \neq j \leq m$. Let $X_i^o = X_i \setminus (\{\tau_1, \dots, \tau_N\} \cap X_i)$ for $0 \leq i \leq m$. Note that τ_1, \dots, τ_N will be used to denote the points of T , X' and various components of X' as well, but this should not lead to any confusion.

Let $T^o = T \setminus \{\tau_1, \dots, \tau_N\}$ and \tilde{T}^o be the formal scheme obtained by the completion of T^o along the closed fiber ($t = 0$) (i.e. the (t) -adic completion). Let $T_i^o = X_i^o \times_k \text{Spec}(k[[t]])$ and \tilde{T}_i^o be the (t) -adic completion of T_i^o (i.e. along X_i^o) for $i = 0, \dots, m$. Since the closed fiber of T^o is the disjoint union of $X_0^o, X_1^o, \dots, X_m^o$,

$$\tilde{T}^o = \tilde{T}_0^o \cup \tilde{T}_1^o \cup \dots \cup \tilde{T}_m^o$$

By base change of Φ_T to \tilde{T}^o we obtain a G -cover $\Phi_{\tilde{T}^o} : \tilde{V}_{T^o} \rightarrow \tilde{T}^o$ and hence a G -cover of the component \tilde{T}_0^o which we will denote by $\Phi_{T_0} : \tilde{V}_0 \rightarrow \tilde{T}_0^o$. Note that Φ_T restricted to the closed fiber is the G -cover $\Phi' : V'_X \rightarrow X'$. Since X_i , for $i = 1, \dots, m$, are the trivial components of the cover $\Phi' : V'_X \rightarrow X'$, the G -covers of the components \tilde{T}_i^o are induced from the trivial cover. Let $\Phi_{T_i} : \tilde{V}_i \rightarrow \tilde{T}_i^o$ be the H_i -cover obtained by pulling back $\Phi_{X_i} : W_{X_i} \rightarrow X_i$ along the composition of morphisms $\tilde{T}_i^o \rightarrow T_i^o \rightarrow X_i^o \rightarrow X_i$, for $i = 1, \dots, m$. Let $\Phi^o : \text{Ind}_G^\Gamma \tilde{V}_0 \cup \text{Ind}_{H_1}^\Gamma \tilde{V}_1 \cup \dots \cup \text{Ind}_{H_m}^\Gamma \tilde{V}_m \rightarrow \tilde{T}^o$ be the Γ -cover of \tilde{T}^o obtained from Φ_{T_i} for $i = 0, \dots, m$.

Let $\hat{T}_{\tau_j} = \text{Spec}(\hat{\mathcal{O}}_{T, \tau_j})$ be the formal neighbourhood of τ_j in T and $\hat{\mathcal{K}}_j$ be the (t) -adic completion of $\tilde{T}_{\tau_j} \setminus \{\tau_j\}$ for $1 \leq j \leq N$, i.e. $\hat{\mathcal{K}}_j$ is the (t) -adic completion of the punctured formal neighbourhood of τ_j in T . We have a natural morphism $\hat{\mathcal{K}}_j \rightarrow \tilde{T}^o$. Note that $\Phi' : V'_X \rightarrow X'$ is étale at $\tau_1, \dots, \tau_N \in X'$, since τ_1, \dots, τ_N lie in X_i for some $1 \leq i \leq m$ and over X_i , Φ' is induced from a trivial cover. Since Φ' is the restriction of Φ_T to the closed fiber, Φ_T is étale over the points τ_1, \dots, τ_N in T . Also note that Φ_{T_i} is the pull back of the H -cover $\Phi_{X_i} : W_{X_i} \rightarrow X_i$ which is étale over τ_1, \dots, τ_N . Hence the pull back of Φ^o to $\hat{\mathcal{K}}_j$ is the Γ -cover of $\hat{\mathcal{K}}_j$ induced from the trivial cover for $1 \leq j \leq N$.

Apply Theorem 6 to obtain a Γ -cover $\Psi : W \rightarrow T$ which induces the Γ -cover Φ^o of \tilde{T}^o and trivial Γ -cover over $\cup_{j=1}^N \hat{T}_{\tau_j}$. The cover W is connected because Γ is generated by G, H_1, \dots, H_m and Γ acts transitively on $\text{Ind}_G^\Gamma \tilde{V}_0 \cup \text{Ind}_{H_1}^\Gamma \tilde{V}_1 \cup \dots \cup \text{Ind}_{H_m}^\Gamma \tilde{V}_m$.

Recall that B is the union of the images of the sections $p_1, \dots, p_r : S \rightarrow X_S$. The branch locus of Ψ is clearly contained in T^o because in the formal neighbourhood of τ_i 's, Ψ restricts to the Γ -cover induced from the trivial cover. Since the pullback of Ψ to \tilde{T}^o is Φ^o , the branch locus of Φ^o maps to the branch locus of Ψ under the morphism $\tilde{T}^o \rightarrow T^o$. Note that for $i = 1, \dots, m$, Φ_{T_i} is étale away from $B_i \times_{X_i^o} \tilde{T}_i^o$ in \tilde{T}_i^o where B_i is the set of r_i smooth marked points of X' lying on X_i . Moreover, the image of $B_i \times_{X_i^o} \tilde{T}_i^o$ under the morphism $\tilde{T}_i^o \rightarrow T$ is contained in $B \times_{X_S} T$.

Note that $\Phi_{T_0} : \tilde{V}_0 \rightarrow \tilde{T}_0^o$ is the pullback of the G -cover $\Phi_S : V_S \rightarrow X_S$ under the morphism $\tilde{T}_0^o \rightarrow \tilde{T}^o \rightarrow T^o \rightarrow T \rightarrow X_S$. Also the branch locus of Φ_S is contained in B . So combining all these we see that Ψ is étale away from $B \times_{X_S} T$.

Finally if $\Gamma = H \rtimes G$ then H is a normal subgroup of Γ and by quotienting one obtains a G -cover $W/H \rightarrow T$. Since $H_1, \dots, H_m \leq H$ the pullback of W/H on \tilde{T}^o is \tilde{V}_{T^o} . Also pullback over \hat{T}_{τ_j} of W/H is a trivial cover for $1 \leq j \leq N$. Hence by Theorem 6 the G -covers $W/H \rightarrow T$ and $V_T \rightarrow T$ are isomorphic. So Ψ dominates Φ_T . \square

4. LEFSCHETZ-ABHYANKAR RESULT

Now we can use a Lefschetz type principle and Abhyankar's lemma to obtain a Γ -cover of X étale over C and dominating the given G -cover. This is a deformation argument similar to [HS09, Proposition 4].

Proposition 13. *In the context of Proposition 12, there exist a Γ -cover $W_s \rightarrow X$ dominating the G -cover $\Phi : V_X \rightarrow X$ which is étale over C .*

Proof. From the conclusion of Proposition 12, there is a Γ -cover $\Psi : W \rightarrow T$ of $\hat{S}_{s'}$ -curves dominating $\Phi_T : V_T \rightarrow T$ where $V_T = V_S \times_S \hat{S}_{s'}$. As in the proof

of Proposition 12, let t be local coordinate of S at s' so that $\hat{S}_{s'} = \text{Spec}(k[[t]])$. By hypothesis $T = X_S \times_S \hat{S}_{s'}$, $V_S \rightarrow X_S$ is a G -cover and its fiber at $s \in S$ is the G -cover $\Phi : V_X \rightarrow X$. Also Ψ is étale away from $B \times_{X_S} T$. Since Ψ is a finite morphism, there exist a finitely generated \mathcal{O}_S -algebra $R \subset k[[t]]$ such that the morphisms $W \rightarrow V_T \rightarrow T$ descend to the morphisms of $S_0 = \text{Spec}(R)$ -curves $W_{S_0} \rightarrow V_{S_0} \rightarrow T_{S_0}$. Note that the morphism $\hat{S}_{s'} \rightarrow S$ is the composition of the structure morphism $\pi : S_0 \rightarrow S$ and the morphism $\hat{S}_{s'} \rightarrow S_0$ induced by the inclusion $R \subset k[[t]]$. So $T_{S_0} = X_S \times_S S_0$ and $V_{S_0} = V_S \times_S S_0$, since T and V_T were base change of X_S and V_S respectively to $\hat{S}_{s'}$. By shrinking S_0 we may assume that S_0 is smooth and for every point $\tau \in S_0$ the fiber of the cover $W_{S_0} \rightarrow T_{S_0}$ is a smooth irreducible Γ -cover $W_\tau \rightarrow X_{\pi(\tau)}$ which is étale away from $\{p_1(\pi(\tau)), \dots, p_r(\pi(\tau))\}$ and dominates the G -cover $V_\tau \rightarrow X_{\pi(\tau)}$.

Since S_0 is an affine S -scheme, we choose an embedding $S_0 \hookrightarrow \mathbb{A}_S^n$, define \bar{S}_0 to be the closure of S_0 in \mathbb{P}_S^n and let $\bar{\pi} : \bar{S}_0 \rightarrow S$ be the structure morphism. Since $\bar{\pi}$ is dominating and projective, it is surjective onto S . Let $T_{\bar{S}_0} = X_S \times_S \bar{S}_0$ and $V_{\bar{S}_0} = V_S \times_S \bar{S}_0$. Note that $T_{\bar{S}_0} \rightarrow \bar{S}_0$ and $V_{\bar{S}_0} \rightarrow \bar{S}_0$ extend $T_{S_0} \rightarrow S_0$ and $V_{S_0} \rightarrow S_0$ respectively.

Let $W_{\bar{S}_0}$ be the normalization of $V_{\bar{S}_0}$ in $k(W_{S_0})$. Note that $W_{S_0} \rightarrow V_{S_0} \rightarrow T_{S_0}$ are finite morphism of normal varieties, so it is the restriction of $W_{\bar{S}_0} \rightarrow V_{\bar{S}_0} \rightarrow T_{\bar{S}_0}$. We summarize the setup in the following diagram.

$$\begin{array}{ccccccc}
 & & W & \longrightarrow & W_{S_0} & \longrightarrow & W_{\bar{S}_0} \\
 & & \downarrow & & \downarrow & & \downarrow \\
 V_S & \longleftarrow & V_T & \longrightarrow & V_{S_0} & \longrightarrow & V_{\bar{S}_0} \\
 \downarrow & & \downarrow & & \downarrow & & \downarrow \\
 X_S & \longleftarrow & T & \longrightarrow & T_{S_0} & \longrightarrow & T_{\bar{S}_0} \\
 \downarrow & & \downarrow & & \downarrow & & \downarrow \\
 S & \longleftarrow & \hat{S}_{s'} & \longrightarrow & S_0 & \longrightarrow & \bar{S}_0
 \end{array}$$

Let $\tau_S \in \bar{S}_0$ be such that $\bar{\pi}(\tau_S) = s$. Let \bar{A}_0 be a curve in \bar{S}_0 passing through τ_S such that $A_0 = \bar{A}_0 \cap S_0$ is non empty. Note that if $\tau_S \in S_0$ then the result follows from the assumptions on S_0 .

Replacing \bar{A}_0 by an open neighbourhood of τ_S , we may assume that the fiber at all points of \bar{A}_0 of the morphism $V_{\bar{A}_0} \rightarrow \bar{A}_0$ are smooth irreducible curves. Let A and \bar{A} be the normalization of A_0 and \bar{A}_0 respectively. Let $\tau_A \in \bar{A}$ be a point lying above $\tau_S \in \bar{A}_0$.

Let B_1 be the finitely many points in $\bar{A} \setminus A$. Let $b_i : \bar{A} \rightarrow V_{\bar{A}}$ be the section obtained by the pull-back of $p_i : S \rightarrow V_S$ along the composition of the morphisms $\bar{A} \rightarrow \bar{A}_0 \rightarrow \bar{S}_0 \rightarrow S$. Let $B_2 = \cup_{i=1}^r \text{im}(b_i)$ and B_V be the union of fibers $\cup_{\zeta \in B_1} V_\zeta$. Note that $W_{\bar{A}} \rightarrow V_{\bar{A}}$ is an H -cover étale away from $B_2 \cup B_V$. Since H is a prime-to- p group, the least common multiple m of the ramification indices at the generic points of B_V is coprime to p . Let $A' \rightarrow \bar{A}$ be a cyclic branched cover totally ramified at the points in B_1 with ramification indices m . Let $W_{A'} \rightarrow V_{A'}$ be the pull back of $W_{\bar{A}} \rightarrow V_{\bar{A}}$. Then applying Abhyankar's lemma we conclude that $W_{A'} \rightarrow V_{A'}$ is étale away from $B_2 \cup B_V$ and it is unramified at the generic points of B_V . Since

$W_{A'} \rightarrow V_{A'}$ is a finite morphism of normal varieties, the purity of Branch locus implies that the morphism is étale away from B_2 .

Let $\tau_{A'} \in A'$ be a point lying above τ_A . The fiber over $\tau_{A'}$ of covering $V_{A'} \rightarrow T_{A'}$ is $V_s \rightarrow T_s$. But this is same as $V_X \rightarrow X$. Since $A' \rightarrow \bar{A}$ is proper and all the fibers of $V_{\bar{A}_0} \rightarrow \bar{A}_0$ are smooth irreducible curves, same is true for the fibers of $V_{A'} \rightarrow A'$.

Let $b'_i : A' \rightarrow V_{A'}$ be the sections obtained by the pullback of b_i along $A' \rightarrow A$. After shrinking A' to an open neighborhood of $\tau_{A'}$ if necessary, we may assume that $\text{im}(b_i)$ and $\text{im}(b_j)$ are disjoint for $i \neq j$. In particular, the branch locus of the cover $W_{A'} \rightarrow V_{A'}$ is smooth. Moreover, being a prime-to- p cover, it is étale locally a Kummer cover. Hence the fiber over every point $\tau \in A'$ of the cover $W_{A'} \rightarrow V_{A'}$ is a cover of smooth curves. Since the fibers of $W_{A'} \rightarrow A'$ are connected for all but finitely many points of A' , Zariski's connectedness theorem tells us that every fiber of $W_{A'} \rightarrow A'$ must be connected. In particular, $W_{\tau_{A'}} \rightarrow T_{\tau_{A'}}$ is a Γ cover of smooth connected curves. This cover dominates $V_{\tau_{A'}} \rightarrow T_{\tau_{A'}}$ which is same as $V_X \rightarrow X$. \square

5. DEGENERATIONS OF COVERS AND SOLVING EMBEDDING PROBLEMS

In this section we use Proposition 12 and 13 to solve certain embedding problems. This is used to obtain more examples of effective subgroups of the fundamental group of a curve for a given embedding problem. The method below combines the technique of “adding branch points” as in [HS09] and “increasing the genus” as in [Ku08, Ku09, BK11].

Theorem 14. *Let C be a smooth affine curve over k and X be the smooth completion of C . Let g be the genus of X and $r = \#(X \setminus C)$. Let $\mathcal{E}(2)$ denote the embedding problem*

$$(2) \quad \begin{array}{ccccccc} & & & \pi_1(C) & & & \\ & & & \downarrow \alpha & & & \\ 1 & \longrightarrow & H & \longrightarrow & \Gamma & \xrightarrow{\phi} & G \longrightarrow 1 \\ & & & & \nearrow & & \downarrow \\ & & & & & & 1. \end{array}$$

Let $\Phi : V_X \rightarrow X$ be the G -cover of X étale over C corresponding to α . Let $B_X = X - C$ and $r = \#(B_X)$ and consider the set B_X to be r marked points on X . Suppose Φ has a degeneration $\Phi' : V'_X \rightarrow X'$ with associated data $(S; \Phi_S : V_S \rightarrow X_S; p_1, \dots, p_r; s; s')$ and let X_1, \dots, X_m be the trivial components of X' . Let r_i be the number of smooth marked points of X' lying on X_i for $i = 1, \dots, m$. Suppose there exist a group homomorphism

$$\theta : \pi_1(X_1 \setminus \{r_1 \text{ points}\}) \times \dots \times \pi_1(X_m \setminus \{r_m \text{ points}\}) \rightarrow H/p(H)$$

such that the image of θ is a relative generating set for $H/p(H)$ in $\Gamma/p(H)$. Then there exist a Γ -cover $W_X \rightarrow X$ which corresponds to a proper solution to $\mathcal{E}(2)$ (i.e. $W_X \rightarrow X$ dominates $\Phi : V_X \rightarrow X$ and is étale over C).

Proof. Let $\bar{\Gamma} = \Gamma/p(H)$, $\bar{H} = H/p(H)$, and $H_i = \theta(\pi_1(X_i \setminus r_i \text{ points}))$ for $i = 1, \dots, m$. Consider the embedding problem $\mathcal{E}(3)$

$$(3) \quad \begin{array}{ccccccc} & & & \pi_1(C) & & & \\ & & & \downarrow \alpha & & & \\ & & \swarrow & & \searrow & & \\ 1 & \longrightarrow & \bar{H} & \longrightarrow & \bar{\Gamma} & \xrightarrow{\phi} & G \longrightarrow 1 \\ & & & & & & \downarrow \\ & & & & & & 1. \end{array}$$

This has prime to p kernel \bar{H} . Since the image of θ is a relative generating set for \bar{H} in $\bar{\Gamma}$, the subgroups H_1, \dots, H_m and G together generate $\bar{\Gamma}$. Moreover, by definition of H_i , there exist an H_i -cover of X_i étale away from r_i points for $1 \leq i \leq m$. So the hypotheses of Proposition 12 and 13 for the embedding problem $\mathcal{E}(3)$ are satisfied. By the conclusion of Proposition 13 there exist a Γ -cover $W_s \rightarrow X$ dominating the G -cover $\Phi : V_X \rightarrow X$ which is étale over C . This is a solution to $\mathcal{E}(3)$.

Since $p(H)$ is a quasi- p group, by [Ha03, Corollary 4.6] or [Pop, Theorem B], the following embedding problem has a solution.

$$\begin{array}{ccccccc} & & & \pi_1(C) & & & \\ & & & \downarrow & & & \\ & & \swarrow & & \searrow & & \\ 1 & \longrightarrow & p(H) & \longrightarrow & \Gamma & \xrightarrow{\phi} & \bar{\Gamma} \longrightarrow 1 \\ & & & & & & \downarrow \\ & & & & & & 1 \end{array}$$

This provides the required Γ -cover. \square

As a consequence of Theorem 14, we obtain the following:

Corollary 15. *Let C be a smooth affine curve over k and X be the smooth completion of C . Let g be the genus of X and $r = \#(X \setminus C)$. Let $\mathcal{E}(2)$ denote the embedding problem above for $\pi_1(C)$ and let μ be the relative rank of H/pH in Γ/pH . Let $\Phi : V_X \rightarrow X$ be the G -cover of X étale over C corresponding to α . Suppose Φ has a degeneration $\Phi' : V'_X \rightarrow X'$ with a trivial component X_1 . Let r_1 be the number of marked points of X' lying on X_1 , $g(X_1)$ be the genus of X_1 and $n_{X_1} = 2g(X_1) + r_1 - 1$. If one of the following holds:*

- (1) $r_1 \geq 1$ and $n_{X_1} \geq \mu$
- (2) $r_1 = 0$ and $g(X_1) \geq \mu$

then there exist a Γ -cover of X dominating Φ which is étale over C .

Proof. Let u_1, \dots, u_μ be the relative generators of H/pH in Γ/pH and H' be the subgroup of H/pH generated by u_1, \dots, u_μ . Note that H' is a prime-to- p group generated by μ elements, so in both scenario (1) and (2) by [SGA1] there exist an epimorphism from $\pi_1(X_1 \setminus \{\text{marked points}\})$ to H' . Hence the corollary follows from Theorem 14. \square

The above result restated in the terminology of effective subgroups becomes the following statement.

Corollary 16. *Let C , X , g , r and μ be as in Corollary 15. Let $\Pi \trianglelefteq \pi_1(C)$ be of finite index such that the embedding problem $\mathcal{E}(2)$ restricts to Π . Let Z be the cover of X étale over C with $\pi_1(Z) = \Pi$ and $\Phi : V_X \times_X Z \rightarrow Z$ be the induced G -cover. Suppose Φ has a degeneration with a trivial component X_1 such that one of the following holds:*

- (1) $r_1 \geq 1$ and $n_{X_1} \geq \mu$
- (2) $r_1 = 0$ and $g(X_1) \geq \mu$

then Π is an effective subgroup for the embedding problem $\mathcal{E}(2)$.

Proof. Note that Φ is a G -cover of Z étale over the preimage of C and it has degeneration with the same properties as in the hypothesis of the above corollary. So using that corollary, we obtain a Γ -cover of Z dominating Φ which is étale over the preimage of C . This Γ -cover provides a solution to the embedding problem $\mathcal{E}(2)$ restricted to Π . Hence Π is an effective subgroup. \square

6. THE CASE OF AFFINE LINE

Let C be the affine line $\mathbb{A}_x^1 = \text{Spec}(k[x])$ and $\mathcal{E} = (\alpha : \pi_1(C) \rightarrow G, \phi : \Gamma \rightarrow G)$ be an embedding problem for $\pi_1(C)$ with $H = \ker \phi$. In [BK11, Theorem 1.3], it was shown that there are infinitely many index p effective subgroups of $\pi_1(\mathbb{A}_x^1)$ for the embedding problem \mathcal{E} .

In [HS09, Theorem 5] (see Proposition 2), it was shown that if the étale cover $D \rightarrow \mathbb{A}_x^1$ is such that the number of points above $x = \infty$ in the smooth completion of D is large enough then $\pi_1(D)$ is an effective subgroup of $\pi_1(C)$ as long as the embedding problem \mathcal{E} restricts to $\pi_1(D)$.

In this section we will demonstrate some sufficient conditions on a subgroup of $\pi_1(\mathbb{A}_x^1)$ to be effective depending only on rank of H and the cover corresponding to α . In fact, we will show that in the collection of all p -cyclic étale covers of high enough genus of \mathbb{A}_x^1 , every member $D \rightarrow \mathbb{A}_x^1$ leads to an effective subgroup $\pi_1(D) \trianglelefteq \pi_1(\mathbb{A}_x^1)$.

Assume Π is an index p subgroup of $\pi_1(C)$. Let $D \rightarrow C$ be the cover corresponding to Π , i.e., D be the normalization of C in $(K^{un})^\Pi$. Since D is an étale p -cyclic cover of the affine line, by Artin-Schrier theory it is given by the equation $z^p - z - f(x)$ for some non-constant polynomial $f(x) \in k[x]$. Let r be the degree of $f(x)$. By changing $f(x)$ if necessary we may assume r is prime to p . Let $Z \rightarrow X$ be the corresponding morphism between their smooth completions. The genus of Z is $g_Z = (p-1)(r-1)/2$ and Z is totally ramified at infinity. So $n_D = 2g_Z + r_D - 1 = 2g_Z = (p-1)(r-1)$.

Let $\mathcal{E} = (\alpha : \pi_1(C) \rightarrow G, \phi : \Gamma \rightarrow G)$ be an embedding problem for $\pi_1(C)$ with $H = \ker \phi$. As observed in the introduction if n_D is less than the rank of $\Gamma/p(\Gamma)$ then Π can not be effective. But $n_D \geq \text{rank}(\Gamma/p(\Gamma))$ is certainly not a sufficient condition for $\Pi = \pi_1(D)$ to be effective, as indicated by the following example.

Example. Let $C = \mathbb{A}^1$, Γ a quasi- p group, $G = \mathbb{Z}/p\mathbb{Z}$ and H a nontrivial prime-to- p group. Note that $\text{rank}(\Gamma/p\Gamma) = 0$. Suppose the map $\alpha : \pi_1(C) \rightarrow G$ be induced by a p -cyclic étale cover $V_C \rightarrow C$ where V_C is also isomorphic to \mathbb{A}^1 . Let $D \rightarrow C$ be any p -cyclic étale cover linearly disjoint from $V_C \rightarrow C$ and $U = D \times_C V_C$, then the embedding problem \mathcal{E} restricts to $\pi_1(D) \trianglelefteq \pi_1(C)$ but if $n_U < \text{rank}(H/p(H))$ then the embedding problem \mathcal{E} restricted to $\pi_1(D)$ has no solution. This is because the

existence of a solution to the embedding problem implies that $H/p(H)$ is a quotient of $\pi_1(U)$. But this is impossible if $n_U < \text{rank}(H/p(H))$.

Though we shall see that if $C = \mathbb{A}^1$, $D \rightarrow C$ is p -cyclic étale and $n_D \geq 2 \text{rank}(H/p(H))$ then $\pi_1(D)$ is indeed an effective subgroup for the embedding problem \mathcal{E} in many cases.

Proposition 17. *Let $\mathcal{E}(4)$ be the embedding problem*

$$(4) \quad \begin{array}{ccccccc} & & & \pi_1(\mathbb{A}^1) & & & \\ & & & \downarrow \alpha & & & \\ & & \swarrow \phi & & \searrow & & \\ 1 & \longrightarrow & H & \longrightarrow & \Gamma & \longrightarrow & G \longrightarrow 1 \\ & & & & & & \downarrow \\ & & & & & & 1 \end{array}$$

Let $V \rightarrow \mathbb{A}_x^1$ be the G -Galois cover corresponding to α and $V_X \rightarrow \mathbb{P}_x^1 = X$ be the morphism corresponding to the smooth completion. Let g be such that there is a homomorphism θ from the surface group Π_g to $H/p(H)$ with the property that $\text{im}(\theta)$ is a relative generating set for $H/p(H)$ in $\Gamma/p(H)$. Let $D \rightarrow \mathbb{A}_x^1$ be an étale p -cyclic cover such that the genus of the smooth completion Z of D is at least g . Let V_Z be the normalization of $V_X \times_X Z$. Suppose that the genus of the normalization of $Z' \times_X V_X$ is same as $g(V_Z)$ for all but finitely p -cyclic covers $Z' \rightarrow X$ branched only at $x = \infty$ with the genus of Z' same as $g(Z)$. Then $\pi_1(D)$ is an effective subgroup for the embedding problem $\mathcal{E}(4)$.

Proof. Since $D \rightarrow \mathbb{A}_x^1$ is p -cyclic étale cover, it is given by the equation $Z^p - Z - (a_r x^r + a_{r-1} x^{r-1} + \dots + a_1 x)$ where r is coprime to p , $a_r \neq 0$ and $a_{ip} = 0$ for $0 \leq i \leq [r/p]$ ([Pri]). Let $A = k[t_1, \dots, t_r]$ where $t_{ip} = 0$ for $1 \leq i \leq [r/p]$ and t_j are indeterminates if p does not divide j . Let $S = \text{Spec}(A)$ and $S^o = S \setminus \{t_r = 0\}$. Let $X_S = X \times_k S$ and Y_S be the normal cover of $\mathbb{P}_y^1 \times_k S$ given by $z^p - z - f(y^{-1})$ where

$$(5) \quad f(w) = w^r + t_{r-1} w^{r-1} + \dots t_1 w^1$$

Note that the genus of the normalization of the fiber of $Y_S \rightarrow S$ for any point of S is constant and Y_S is normal. Hence every fiber of Y_S over a closed point of S is a smooth curve (Lemma 11). Also note that the cover $Y_S \rightarrow \mathbb{P}_y^1 \times_k S$ is branched only at $y = 0$ in $\mathbb{P}_y^1 \times S$ because the discriminant is -1 on $\mathbb{A}_{y^{-1}}^1 \times S$. Let $Y \rightarrow \mathbb{P}_y^1$ be the fiber of $Y_S \rightarrow \mathbb{P}_y^1 \times_k S$ at the point $(t_1 = 0, \dots, t_r = 0) \in S$ and note that $y = 0$ defines a unique point τ in Y .

Let F be the locus of $t_r - xy = 0$ in $\mathbb{P}_x^1 \times_k \mathbb{P}_y^1 \times_k S$ and $Y_F = Y_S \times_{\mathbb{P}_y^1 \times_k S} F$. Let $X_F = X_S \times_{\mathbb{P}_x^1 \times_k S} F$ and T be the fiber product $X_F \times_F Y_F$. Note that $X_F = F$ and $T = Y_F$. By definition, $\Psi_X : V_X \rightarrow X$ is a G -cover of X étale over \mathbb{A}_x^1 .

$$\begin{array}{ccccc}
& & T = Y_F & & \\
& \swarrow & & \searrow & \\
& X_F = F & & Y_S & \\
& \swarrow \downarrow \searrow & & & \\
V_X & X_S = \mathbb{P}_x^1 \times S & \longleftarrow \mathbb{P}_x^1 \times \mathbb{P}_y^1 \times S & \longrightarrow \mathbb{P}_y^1 \times S & \\
\downarrow & \swarrow & & & \\
X = \mathbb{P}_x^1 & & & &
\end{array}$$

The morphism $T \rightarrow X_F$ is a family of covers parametrized by S . Let $s' \in S$ be the point $(t_1 = 0, \dots, t_r = 0)$ and s be any point in S^o . Note that the morphism $F \rightarrow \mathbb{P}_y^1 \times S$ is an isomorphism away from $t_r = 0$. So the fiber $T_s = Y_{S,s}$ is smooth. Let $T_{s'} \rightarrow X_{s'}$ be the fiber of $T \rightarrow X_F$ over s' . Then $X_{s'}$ is the union of $X = \mathbb{P}_x^1$ and \mathbb{P}_y^1 intersecting transversally at $(x = 0, y = 0)$ and $T_{s'}$ is the union $X = \mathbb{P}_x^1$ and Y intersecting transversally at the point $(x = 0, \tau)$. The fiber over s , $T_s \rightarrow X_s$ is a p -cyclic cover of smooth curves. Since at $s \in S^o$, $t_r \neq 0$, the projection map $X_F \rightarrow X$ restricted to X_s is an isomorphism. Hence $X_s = \mathbb{P}_x^1 = X$. Moreover, if s is the point $(t_1 = b_1, \dots, t_r = b_r)$, then the cover T_s is locally given by the equation

$$Z^p - Z - \left(\frac{1}{b_r^r} x^r + \frac{b_{r-1}}{b_r^{r-1}} x^{r-1} + \dots + \frac{b_1}{b_r} x \right)$$

because $y^{-1} = xt_r^{-1}$ on T .

Let $\Phi_T : V_T \rightarrow T$ be the normalized pullback of $V_X \rightarrow X$ to T along the morphism $T \rightarrow X_F \rightarrow X_S \rightarrow X$. So $V_T \rightarrow T$ is a G -cover. Let

$$S^1 = \{s \in S^o : \text{the genus of the normalization of } V_X \times_X T_s = g(V_Z)\}.$$

By hypothesis, S^1 is an open dense subset of S^o . The normalization of the fiber $V_{T,s}$ of V_T at $s \in S^1$ is the normalization of $V_X \times_X T_s$. The genus of the normalization of $V_{T,s}$ for all $s \in S^1$ is constant. Moreover V_T is normal, hence $V_{T,s}$ is smooth for all $s \in S^1$ (Lemma 11). Hence Φ_s is a cover of smooth irreducible curves dominating V_X for all $s \in S^1$. Let the fiber of Φ_T at s' be denoted by the morphism $V_{s'} \rightarrow T_{s'}$. Since $V_X \rightarrow X$ is étale at $x = 0$ and $T_{s'}$ is the union of Y and X intersecting only at $x = 0$ in X and τ in Y , the fiber $V_{s'}$ is the union of V_X and $|G|$ copies of Y which intersect in V_X at the $|G|$ preimages of $x = 0$ and in Y at $y = 0$. In particular for any $s \in S^1$, $\Phi_{s'}$ is a degeneration of Φ_s and the irreducible component Y of $T_{s'}$ is a trivial component of $\Phi_{s'}$.

Finally choosing b_i , and hence $s \in S^o$, appropriately we may assume $T_s \rightarrow \mathbb{P}_x^1$ is the same cover as $Z \rightarrow \mathbb{P}_x^1$. More precisely, let $b_r = (a_r)^{-1/r}$, $b_i = a_i b_r^i$ for $1 \leq i \leq r-1$ then the local equation of T_s is same as that of D . Hence $k(T_s) = k(D)$ and the genus $g(T_s) = g(Y)$ is at least g . Also note that $s \in S^1$.

Note that by [SGA1] there exist an epimorphism from $\pi_1(Y)$ to the prime to p part of Π_g . Composing this with θ and noting that $H/p(H)$ is a prime to p group, we obtain a homomorphism $\tilde{\theta} : \pi_1(Y) \rightarrow H/p(H)$ such that $\text{im}(\tilde{\theta})$ is a relative

generating set for $H/p(H)$ in $\Gamma/p(H)$. So applying Theorem 14, we obtain a Γ -cover of T_s which dominates the G -cover $V_X \times_X T_s \rightarrow T_s$. Hence $\pi_1(D)$ is an effective subgroup of $\pi_1(\mathbb{A}_x^1)$ for the embedding problem $\mathcal{E}(4)$. \square

For a Galois cover $U \rightarrow V$ of smooth connected k -curves, by the upper jumps at a branch point $v \in V$ we mean the upper jumps of the ramification filtration of the local field extension $\hat{\mathcal{O}}_{U,u}/\hat{\mathcal{O}}_{V,v}$ for any point $u \in U$ lying above v . Note that the set upper jumps does not depend upon the choice of $u \in U$ lying above $v \in V$ since $U \rightarrow V$ is a Galois cover.

Corollary 18. *Let $\mathcal{E}(4)$ be the embedding problem in Proposition 17, $V_X \rightarrow \mathbb{P}_x^1$ be the G -Galois cover corresponding to α and g be as in Proposition 17. Let $D \rightarrow \mathbb{A}_x^1$ be an étale p -cyclic cover such that the genus of the smooth completion Z of D is at least g and the upper jump of the cover $Z \rightarrow \mathbb{P}_x^1$ at $x = \infty$ is different from all the upper jumps of $V_X \rightarrow \mathbb{P}_x^1$ at $x = \infty$. Then $\pi_1(D)$ is an effective subgroup for the embedding problem $\mathcal{E}(4)$.*

Proof. Note that $Z \rightarrow \mathbb{P}_x^1$ being a p -cyclic cover is totally ramified at $x = \infty$. Let $\tau \in Z$ be the point lying above $x = \infty$. Let $\beta_1, \dots, \beta_l \in V_X$ be the points lying above $x = \infty$ under the morphism $V_X \rightarrow \mathbb{P}_x^1$. Let $I_j \leq G$ be the inertia group of $V_X \rightarrow \mathbb{P}_x^1$ at β_j for $1 \leq j \leq l$. Since $V_X \rightarrow \mathbb{P}_x^1$ is a Galois cover all the I_j 's are conjugates of each other. Also we know that the degree of the morphism $V_X \rightarrow \mathbb{P}_x^1$ is $|G|$ and the ramification index at β_j is $|I_j| = e$ (say). So $|G| = el$.

Let \hat{R} be the completion of the stalk of V_X at β_1 , K the fraction field of \hat{R} , \hat{S} be the completion of the stalk of Z at τ and L the fraction field of \hat{S} . Then $\text{Gal}(K/k((x^{-1}))) = I_1$, $\text{Gal}(L/k((x^{-1}))) = \mathbb{Z}/p\mathbb{Z}$. Since the upper jumps of the two local extensions are distinct, K and L are linearly disjoint over $k((x^{-1}))$. Hence $k(V_X)$ and $k(Z)$ are linearly disjoint over $k(x)$. So $V_X \times_{\mathbb{P}_x^1} Z$ is an irreducible curve with the function field being the compositum $k(V_X)k(Z)$. Let V_Z be the normalization of $V_X \times_{\mathbb{P}_x^1} Z$. Then $V_Z \rightarrow \mathbb{P}_x^1$ is a $G \times \mathbb{Z}/p\mathbb{Z}$ -Galois cover branched only at ∞ . At a point in V_Z lying above $(\beta_1, \tau) \in V_X \times_{\mathbb{P}_x^1} Z$, the ramification index of the cover $V_Z \rightarrow \mathbb{P}_x^1$, which can be computed by passing to the completion of stalks at β_1 of V_X and τ of Z , comes out to be the degree of the field extension $[KL : k((x^{-1}))]$. But this degree equals pe , since K and L are linearly disjoint, $[K : k((x^{-1}))] = e$ and $[L : k((x^{-1}))] = p$. Also the degree of the morphism $V_Z \rightarrow \mathbb{P}_x^1$ is $p|G| = pel$. Since $V_Z \rightarrow \mathbb{P}_x^1$ is a Galois cover, there are exactly l points in V_Z over $x = \infty$ one each lying above $(\beta_j, \tau) \in V_X \times_{\mathbb{P}_x^1} Z$.

Let u_1, \dots, u_a be the upper jumps of the ramification filtration on the inertia group I_1 of the cover $V_X \rightarrow \mathbb{P}_x^1$ and r be the upper jump of the inertia group $\mathbb{Z}/p\mathbb{Z}$ at τ of the cover $Z \rightarrow \mathbb{P}_x^1$. Since r is different from u_1, \dots, u_a by [Ku014, Corollary 2.5] the ramification filtration is completely determined by the ramification filtration on I_1 and r . Let $Z' \rightarrow \mathbb{P}_x^1$ be another p -cyclic cover branched only at $x = \infty$ and the genus of Z' is $g(Z)$. Then the upper jump at $x = \infty$ of $Z' \rightarrow \mathbb{P}_x^1$ is also r (since $g(Z) = (p-1)(r-1)/2$ depends only on r). Let $V_{Z'}$ be the normalization of $V_X \times_{\mathbb{P}_x^1} Z'$. We observe that like $V_Z \rightarrow \mathbb{P}_x^1$, the cover $V_{Z'} \rightarrow \mathbb{P}_x^1$ is branched only at $x = \infty$, like in V_Z , there are exactly l points in $V_{Z'}$ lying above $x = \infty$ and the ramification filtration at these l points are same as the ramification filtration on the l points in V_Z lying above $x = \infty$. Since the degree and the ramification behaviour of $V_Z \rightarrow \mathbb{P}_x^1$ and $V_{Z'} \rightarrow \mathbb{P}_x^1$ are same, by Riemann-Hurwitz formula and

Hilbert's different formula, the genus $g(V_Z) = g(V_{Z'})$. The result now follows from Proposition 17. \square

Corollary 19. *Let $\mathcal{E}(4)$ be the embedding problem and $V \rightarrow \mathbb{A}_x^1$ be the G -Galois cover corresponding to α as in Proposition 17. Let $Z \rightarrow \mathbb{P}_x^1$ be a p -cyclic cover branched only at $x = \infty$ such that $g = g(Z)$ is at least the relative rank of $H/p(H)$ is $\Gamma/p(H)$ and the upper jump of the cover $Z \rightarrow \mathbb{P}_x^1$ at $x = \infty$ is different from all the upper jumps of $V_X \rightarrow \mathbb{P}_x^1$ at $x = \infty$. Let $D \subset Z$ be the complement of points lying above $x = \infty$. Then $\pi_1(D)$ is an effective subgroup for the embedding problem $\mathcal{E}(4)$.*

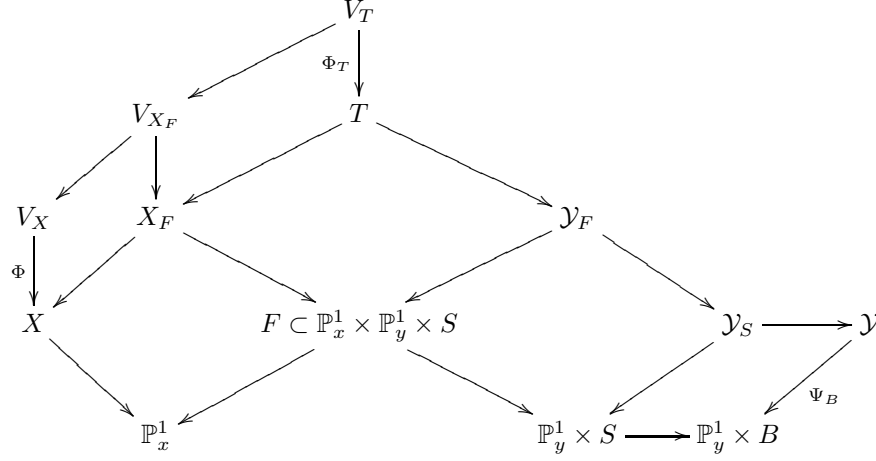
Proof. Let $\{a_1, \dots, a_g\} \subset H/p(H)$ be a relative generating set for $H/p(H)$ in $\Gamma/p(H)$. Note that Π_g is the quotient of the free group on $2g$ generators $A_1, \dots, A_g, B_1, \dots, B_g$ by the subgroup generated by $[A_1, B_1] \cdot [A_2, B_2] \cdots [A_g, B_g]$. So there exist a homomorphism from Π_g to $H/p(H)$ which takes $A_i \rightarrow a_i$ and B_i to identity. Hence there exist $\theta : \Pi_g \rightarrow H/p(H)$ such that image of θ is a relative generating set for $H/p(H)$ in $\Gamma/p(H)$. The result now follows from the above corollary. \square

Remark 20. An étale p -cyclic cover $D \rightarrow \mathbb{A}_x^1$ is given by the polynomial equation $z^p - z = f(x)$ where $f(x)$ is polynomial of degree r coprime to p . For such a cover the upper jump of the inertia group $\mathbb{Z}/p\mathbb{Z}$ at $x = \infty$ is r and genus of the smooth completion Z of D is $(p-1)(r-1)/2$. So given an étale cover $V \rightarrow \mathbb{A}_x^1$, the hypothesis on the cover $Z \rightarrow \mathbb{P}_x^1$ of the above two corollaries will hold for all but finitely many values of r .

7. EXISTENCE OF DEGENERATIONS

We will give a few more examples below where degenerations exist.

Let X be a smooth irreducible projective curve over k with a nonempty set of marked points. Let $C = X \setminus \{\text{the marked points}\}$ and $\Theta : X \rightarrow \mathbb{P}_x^1$ be a finite surjective generically separable morphism such that $\Theta^{-1}(x = \infty)$ is any given nonempty subset of $X \setminus C$ and Θ is étale over $x = 0$. Such a Θ exist by Lemma 7. Let m be the degree of the morphism Θ and $\{r_1, \dots, r_m\} = \Theta^{-1}(x = 0)$. Let $\Phi : V_X \rightarrow X$ be a G -cover étale over C . For a smooth k -variety B , let $\Psi_B : \mathcal{Y} \rightarrow \mathbb{P}_y^1 \times B$ be a family of smooth covers of \mathbb{P}_y^1 ramified only at $y = 0$ and let m' be the number of points in each fiber $\Psi_b : \mathcal{Y}_b \rightarrow \mathbb{P}_y^1$ lying above $y = 0$, for all $b \in B$. Let $S = \mathbb{A}_t^1 \times B$ and let F be the closure of the zero locus of $t - xy$ in $\mathbb{P}_x^1 \times \mathbb{P}_y^1 \times S$. Let $X_F = X \times_{\mathbb{P}_x^1} F$, $V_{X_F} = V_X \times_X X_F = V_X \times_{\mathbb{P}_x^1} F$, $\mathcal{Y}_S = \mathcal{Y} \times \mathbb{A}_t^1 = \mathcal{Y} \times_{\mathbb{P}_y^1 \times B} (\mathbb{P}_y^1 \times S)$, $\mathcal{Y}_F = \mathcal{Y}_S \times_{\mathbb{P}_y^1 \times S} F = \mathcal{Y} \times_{\mathbb{P}_y^1 \times B} F$, T be the normalization of $X_F \times_F \mathcal{Y}_F$ and V_T be the normalization of $V_{X_F} \times_{X_F} T$ which is same as the normalization of $V_{X_F} \times_F \mathcal{Y}_F$. Observe that the normalized base change of $\Phi : V_X \rightarrow X$ via the morphism $T \rightarrow X_F \rightarrow X$, is a G -cover of S -curves $\Phi_T : V_T \rightarrow T$.



Let $\Phi' : V' \rightarrow T'$ be the G -cover of B -curves obtained by looking at the fiber of $\Phi_T : V_T \rightarrow T$ over $t = 0$. For $b \in B$, let $\Phi'_b : V_{0,b} \rightarrow T_{0,b}$ denote the fiber of Φ' at b .

Proposition 21. *Let the setup be as above. For a closed point $s = (a, b) \in S = \mathbb{A}_t^1 \times B$, let Φ_s be the fiber of Φ_T at s . The set*

$$U = \{s \in S : \Phi_s \text{ is a } G\text{-cover of smooth irreducible curves}\}$$

is a nonempty open subset of S . Moreover for $s = (a, b) \in U$, Φ'_b is a degeneration of Φ_s and the fiber of T at $(t = 0, b)$ $T_{0,b}$ consist of m copies of \mathcal{Y}_b which are trivial components of Φ'_b . Also the cover $T_s \rightarrow X$ is étale over C .

Proof. The fiber $F_{0,b}$ of $F \rightarrow S$ at $(t = 0, b) \in S$ for any $b \in B$ is \mathbb{P}_x^1 and \mathbb{P}_y^1 intersecting transversally at $x = y = 0$. Let $\Psi_b^{-1}(y = 0) = \{s_1, \dots, s_{m'}\}$. The fiber $T_{0,b}$ of $T \rightarrow S$ at $(0, b)$ consist of m copies of \mathcal{Y}_b and m' copies of X where each copy of X intersect each copy of \mathcal{Y}_b at exactly one point. This can be seen as follows. Let $\bar{T} = X_F \times_F \mathcal{Y}_F$ and $T \rightarrow \bar{T}$ be the normalization morphism.

$$\begin{aligned} \bar{T}_{0,b} &= \bar{T} \times_S (0, b) \\ &= (X_F \times_F \mathcal{Y}_F) \times_S (0, b) \\ &= (X \times_{\mathbb{P}_x^1} F) \times_F (F \times_{\mathbb{P}_y^1 \times S} \mathcal{Y}_S) \times_S (0, b) \\ &= (X \times_{\mathbb{P}_x^1} F) \times_F (F_{0,b} \times_{\mathbb{P}_y^1 \times (0,b)} \mathcal{Y}_b) \\ &= X \times_{\mathbb{P}_x^1} (F \times_F ((\mathbb{P}_x^1 \cup \mathbb{P}_y^1 \text{ intersecting transversally at } x = y = 0) \times_{\mathbb{P}_y^1} \mathcal{Y}_b)) \\ &= X \times_{\mathbb{P}_x^1} (\mathcal{Y}_b \cup m' \text{ copies of } \mathbb{P}_x^1 \text{ where } j^{\text{th}} \text{ copy of } \mathbb{P}_x^1 \text{ intersect } \mathcal{Y}_b \text{ at exactly} \\ &\quad \text{one point } (x = 0, s_j) \in \mathbb{P}_x^1 \times \mathcal{Y}_b, 1 \leq j \leq m') \\ &= m \text{ copies of } \mathcal{Y}_b \text{ and } m' \text{ copies of } X \text{ where the } j^{\text{th}} \text{ copy of } X \text{ intersect the} \\ &\quad i^{\text{th}} \text{ copy of } \mathcal{Y}_b \text{ at exactly one point } (r_i, s_j) \in X \times \mathcal{Y}_b, 1 \leq i \leq m, 1 \leq j \leq m'. \end{aligned}$$

Note that $\mathcal{O}_F = (\mathcal{O}_{\mathbb{P}_x^1} \otimes_k \mathcal{O}_{\mathbb{P}_y^1} \otimes_k k[t] \otimes_k \mathcal{O}_B) / (t - xy)$. So $\mathcal{O}_{\bar{T}} = (\mathcal{O}_X \otimes_k \mathcal{O}_Y \otimes_k k[t]) / (t - xy)$ since \mathcal{O}_X is a flat $\mathcal{O}_{\mathbb{P}_x^1}$ -algebra and \mathcal{O}_Y is a flat $\mathcal{O}_{\mathbb{P}_y^1} \otimes_k \mathcal{O}_B$ -algebra. So \bar{T} is regular at any closed point in $t = 0$ locus. Hence the normalization morphism $T \rightarrow \bar{T}$ is an isomorphism at these points. So $T_{0,b} = \bar{T}_{0,b}$.

Note that Φ_T is the normalized base change of $V_X \rightarrow X$ and the fiber of $T \rightarrow S$ over $(t = 0, b)$ consist of components isomorphic to X and \mathcal{Y}_b . So for the G -cover $\Phi'_b : V_{0,b} \rightarrow T_{0,b}$ each copy of \mathcal{Y}_b is a trivial component of Φ'_b .

Let $\eta = S \times_{\mathbb{A}_t^1} \text{Spec}(k(t)) = B \times \text{Spec}(k(t))$. For any closed point $b \in B$, let $\eta_b = b \times \text{Spec}(k(t))$ denote the corresponding closed point of η .

Claim. Over the point η_b of S , $V_{T,\eta_b} \rightarrow T_{\eta_b}$ is a G -cover of irreducible η_b -curves.

Proof. Note that

$$\begin{aligned} T_{\eta_b} &= T \times_S \eta_b = (\mathcal{Y}_F \times_F X_F) \times_S \eta_b \\ &= (\mathcal{Y}_b \otimes_k k(t)) \times_{F \times_S \eta_b} (X \otimes_k k(t)) \end{aligned}$$

So show that T_{η_b} is irreducible, it is enough to show that the function fields of the covers $\mathcal{Y}_b \otimes_k k(t)$ and $X \otimes_k k(t)$ of $F \times_S \eta_b$ are linearly disjoint over the function field of $F \times_S \eta_b$. Also note that the composition of morphisms $\mathcal{Y}_b \otimes_k k(t) \rightarrow F \times_S \eta_b \xrightarrow{\sim} \mathbb{P}_y^1 \otimes_k k(t)$ is the extension of base field of $\mathcal{Y}_b \rightarrow \mathbb{P}_y^1$ to $k(t)$ and similarly $X \otimes_k k(t) \rightarrow F \times_S \eta_b \xrightarrow{\sim} \mathbb{P}_x^1 \otimes_k k(t)$ is the extension of base field of $X \rightarrow \mathbb{P}_x^1$ to $k(t)$. Moreover the composition of isomorphism $\mathbb{P}_y^1 \otimes_k k(t) \rightarrow F \times_S \eta_b \rightarrow \mathbb{P}_x^1 \otimes_k k(t)$ which we will call Θ is given by $y \mapsto t/x$.

Let $\alpha \in k(\mathcal{Y}_b)$ be such that $k(\mathcal{Y}_b) = k(y)[\alpha]$ and $f(y, Z)$ be the minimal polynomial of α in $k(y)[Z]$. We have a cover $\mathcal{Y}_b \otimes_k k(t) \rightarrow \mathbb{P}_x^1 \otimes_k k(t)$ via the isomorphism Θ . Consider the resulting field extension $L_2 = k(t)(\mathcal{Y}_b)$ of $k(t, x)$. We also have a field extension $L_1 = k(t)(X)$ of $k(t, x)$ obtained from the morphism $X \rightarrow \mathbb{P}_x^1$ base changed to $k(t)$. To see that T_{η_b} is irreducible, it is enough to show that L_1 and L_2 are linearly disjoint over $k(t, x)$. Note that $L_2 \cong k(t, y)[\alpha]$, hence viewing L_2 as an extension $k(t, x)$, we get that $L_2 = k(t, x)[\alpha']$ where α' is a root of the irreducible polynomial $f(t/x, Z)$ in $k(t, x)[Z]$. We observe that L_1 and L_2 are linearly disjoint over $k(t, x)$ iff $[L_1 L_2 : L_1] = [L_2 : k(t, x)] = \deg_Z(f(t/x, Z))$. But $L_1 L_2 = L_1[\alpha']$, so it is enough to show $f(t/x, Z)$ is irreducible in $k(t)(X)[Z]$. Let $f(t/x, Z) = Z^n + a_{n-1}(t/x)Z^{n-1} + \dots + a_0(t/x)$ and nonzero $\gamma \in k$ be such that $x = \gamma$ is not a pole of $a_0(t/x), \dots, a_{n-1}(t/x)$. Let $\tilde{\gamma} \in X$ be a closed point lying above the closed $x = \gamma$ of \mathbb{P}_x^1 . Then at point $\tilde{\gamma}$ the polynomial $f(t/x, Z)$ reduces to $f(t/\gamma, Z) \in k(t)[Z]$. Since $f(y, Z)$ is irreducible in $k(y)[Z]$, $f(t/\gamma, Z)$ is irreducible in $k(t)[Z]$. Hence $f(t/x, Z)$ is also irreducible in $k(t)(X)[Z]$.

The proof of the irreducibility of V_{T,η_b} is also similar and can be obtained by replacing X by V_X in the above argument. \square

From the claim it follows that for any $b \in B$ there exist a nonempty open subset U_b of \mathbb{A}_t^1 such that the fiber of Φ_T over $(a, b) \in S$ for any closed point a of U_b is a G -cover of irreducible curves. Note that T and V_T are normal, hence so is T_η and $V_{T,\eta}$. So most of the fibers of Φ_T is a cover of smooth irreducible curves. Hence U is a nonempty open set.

Finally there is a morphism $T_\eta \rightarrow X_\eta = X \times_k \eta$ coming from the morphism $T \rightarrow X_F$ which is étale away from $y = 0$. But the rational function y on X_η is same as t/x . So $T_\eta \rightarrow X_\eta$ is étale away from points lying over $x = \infty$. In other words, $T_\eta \rightarrow X_\eta$ is étale over $C \times_k \eta$.

So for any closed point (a, b) of $U \subset S = \mathbb{A}_t^1 \times B$, Φ'_b is a degeneration of the G -cover $\Phi_{a,b} : V_{a,b} \rightarrow T_{a,b}$ and $T_{a,b} \rightarrow X$ is a smooth irreducible cover étale over C (of the same degree as $\mathcal{Y}_b \rightarrow \mathbb{P}_y^1$). \square

Corollary 22. *Let $\mathcal{E}(6)$ be the embedding problem*

$$(6) \quad \begin{array}{ccccccc} & & & \pi_1(C) & & & \\ & & & \downarrow \alpha & & & \\ & & \swarrow \phi & & \searrow & & \\ 1 & \longrightarrow & H & \longrightarrow & \Gamma & \longrightarrow & G \longrightarrow 1 \\ & & & & & & \downarrow \\ & & & & & & 1 \end{array}$$

Let X be the smooth completion of C and $\Phi : V_X \rightarrow X$ be a G -cover étale over C corresponding to α . Let the notation and hypothesis be as in Proposition 21. Let $s = (a, b) \in U \subset S$ be a fixed point, $D \subset T_s$ be the preimage of C under the morphism $T_s \rightarrow X$ and g be the genus of \mathcal{Y}_b . Then $\pi_1(D)$ is an effective subgroup of $\pi_1(C)$ for the given embedding problem if there exist a homomorphism $\theta : \Pi_g^m \rightarrow H/p(H)$ with $\text{im}(\theta)$ a relative generating subset of $H/p(H)$ in $\Gamma/p(H)$. Here Π_g is the surface group of genus g .

Proof. By the above proposition Φ_s has a degeneration to Φ'_b with m copies of \mathcal{Y}_b as trivial components. Also prime to p part of $\pi_1(\mathcal{Y}_b)$ is the prime to p part of the profinite completion of Π_g . Hence the corollary follows from Theorem 14. \square

Corollary 23. *Let Π be an index p -normal subgroup of $\pi_1(C)$ and $D \rightarrow C$ be the corresponding étale cover. Consider the embedding problem $\mathcal{E}(6)$ in the above corollary and let $V \rightarrow C$ be the G -Galois cover corresponding to α . Let X , V_X and Z be the smooth completion of C , V and D respectively. Suppose there exist a separable cover $\theta : X \rightarrow \mathbb{P}_x^1$ étale over $x = 0$ with $\theta^{-1}(x = \infty) \cap C$ empty and a p -cyclic cover $Y \rightarrow \mathbb{P}_x^1$ branched only at $x = \infty$ such that the normalization of the cover $X \times_{\mathbb{P}_x^1} Y \rightarrow X$ is same as the cover $Z \rightarrow X$. Also assume that the genus g_Y is at least the relative rank of $H/p(H)$ in $\Gamma/p(H)$ and the upper jump of $Y \rightarrow \mathbb{P}_x^1$ is different from all the upper jumps of $V_X \rightarrow \mathbb{P}_x^1$ at all the points of V_X lying above $x = \infty$. Then Π is an effective subgroup of $\pi_1(C)$ for the embedding problem (6).*

Proof. Since $Y \rightarrow \mathbb{P}_x^1$ is a p -cyclic cover, it is given by an Artin-Schreier polynomial $z^p - z - f(x)$ where $f(x)$ is a polynomial of degree r for some r coprime to p . Let $f(x) = a_r x^r + a_{r-1} x^{r-1} + \dots + a_0$ with $a_r \neq 0$. Let $B = \mathbb{A}^r$ and $\mathcal{Y} \rightarrow B \times \mathbb{P}_y^1$ be the cover given by $z^p - z - (y^{-r} + b_{r-1} y^{-r+1} + \dots + b_0)$ where b_i 's are coordinates of B . Note that this is a family of p -cyclic covers of \mathbb{P}_y^1 branched only at $y = 0$.

Let $S = B \times \mathbb{A}_t^1$, F , T , $\Phi_T : V_T \rightarrow T$, etc. be defined as in the setup before Proposition 21. Let $U = \{s \in S : \Phi_s \text{ is a cover of smooth curves}\}$ also be as in Proposition 21.

Note that Φ_s is the fiber of $\Phi_T : V_T \rightarrow T$ where V_T and T are the normalization of $V_{X_F} \times_F \mathcal{Y}_F$ and $X_F \times_F \mathcal{Y}_F$ respectively. So for $s = (t, b) \in U$ with $b = (\beta_{r-1}, \dots, \beta_0)$, T is the normalization of $X \times_{\mathbb{P}_x^1} \mathcal{Y}_s$ where $\mathcal{Y}_s \rightarrow \mathbb{P}_x^1$ is given by

$$z^p - z - (t^{-r} x^r + \beta_{r-1} t^{-r+1} x^{r-1} \dots + \beta_0).$$

By the hypothesis on the upper jumps of $V_X \rightarrow \mathbb{P}_x^1$ and the upper jump of $Y \rightarrow \mathbb{P}_x^1$ which is same as the upper jump of $\mathcal{Y}_s \rightarrow \mathbb{P}_x^1$, we obtain that for every point $s = (t, b) \in S$ with $t \neq 0$ the normalization of T_s and $V_{T,s}$ have constant genus (i.e. independent of s). Hence by Lemma 11 $U = S \setminus \{t = 0\}$. So by Corollary 22 we conclude that $\pi_1(T_s \setminus \{\text{points lying above } x = \infty\})$ is an effective subgroup

of $\pi_1(C)$ for all $s = (t, b) \in S$ with $t \neq 0$. By an appropriate choice of t and b one can arrange that the cover $T_s \rightarrow X$ is same as $Z \rightarrow X$. Hence $\pi_1(D)$ is an effective subgroup of $\pi_1(C)$. \square

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STATISTICS AND MATHEMATICS UNIT, INDIAN STATISTICAL INSTITUTE,, BANGALORE, INDIA-560059

E-mail address: `manish@isibang.ac.in`